Transforming Mathematics: Using Dynamic Geometry Software to Strengthen Understanding of Enlargement and Similarity

Jo Denton [1] *

[1] Centre for Education Studies, University of Warwick, Coventry, CV4 7AL, UK
* Correspondence: J.L.Denton@warwick.ac.uk

Abstract This article discusses the potential to use Dynamic Geometry Software (DGS) to form conceptual links across enlargement and similarity by developing learners' understanding of scale factor and ratio. From the theoretical perspective of situated abstraction, a combination of both situated cognition and mathematical abstraction, it analyses existing literature on the teaching, learning and assessment of enlargement and similarity as well as literature on DGS and how it acts as a microworld, where an array of situations in a mathematically consistent environment can be created. Particular focus is given to how the dragging and measurement facilities in DGS support abstractions through both amplification and reorganisation of traditional pencil and paper methods. The empirical element of this article describes a small scale classroom based project on the use of DGS as a microworld for transformation geometry. Through analysing learners' dialogue and written responses to tasks, it proposes that a combination of minimally pre-constructed tasks, peer-discussion and utilising the dragging and measurement facilities, can enhance the observation of patterns in transformation geometry and concludes that these conditions can support learners to move from the particular to the general, allowing abstractions to be conceived and strengthening learners' understanding of enlargement and similarity.

Peer review: This article has been subject to a double blind peer review process

Keywords: Dynamic Geometry Software (DGS); Mathematical Abstraction; Microworld; Situated Abstraction; Situated Cognition; Transformation Geometry

GLOSSARY OF TERMS

Situated cognition: the theory that all learning is in context as the classroom can provide the context.

Mathematical abstraction: the theory of moving from particular or specific examples to the ability to generalise.

Situated abstraction: where situated cognition and mathematical abstraction meet; the theory of making connections in mathematics through making sense of a concrete situation.

Microworld: a digital educational environment where learners can explore and receive immediate feedback from the technology.

Amplification: in terms of transformations using DGS, the ability to perform large numbers of transformations which would be time consuming without the technology.

Reorganisation: in terms of transformations using DGS, the ability to see between the usual transformations, for example enlargements between scale factor 2 and scale factor -2, allowing the learner to see what happens at scale factor 1, and when an enlargement begins to reduce, or at what point the image goes in the opposite direction etc.

INTRODUCTION

Dynamic Geometry Software (DGS) is a digital technology which was developed in the 1980s; however it has not had the expected impact in the classroom (Jones, 2011; JMC, 2011). Even in 'successful' mathematics departments, DGS is often primarily used as a demonstration tool (Ruthven *et al.*, 2008) which has limited potential for learners to form and test their own conjectures. From the theoretical perspective of situated abstraction, a combination of both situated cognition and mathematical abstraction, in this article I discuss the potential to use dynamic geometry to form conceptual links across enlargement and similarity by developing learners' understanding of scale factor and ratio. Firstly I analyse the wealth of literature on DGS and how it acts as a microworld, where an array of situations in a mathematically consistent environment can be created. How this applies to the theoretical framework of situated abstraction is then discussed, with particular focus on how the dragging and measurement facilities in DGS support abstractions through both amplification and reorganisation of traditional pencil and paper methods. Literature on the teaching, learning and assessment of enlargement and similarity is reviewed, before presenting and analysing a small scale classroom based project on the use of DGS as a microworld for transformation geometry. Despite the plethora of research on dynamic geometry, "research on using DGS to teach transformation geometry is relatively limited" (Jones, 2012, p.49). Through analysing learners' dialogue and written responses to tasks, I propose that a combination of minimally pre-constructed tasks, peer-discussion and utilising the dragging and measurement facilities, can enhance the observation of patterns in transformation geometry. Finally, I conclude that these conditions can support learners to move from the particular to the general, allowing abstractions to be conceived and strengthening learners' understanding of enlargement and similarity.

LITERATURE REVIEW

Digital Technologies

Digital technologies play three distinct roles in education: tutor, tool and tutee (Taylor, 1980), however Ofsted reports indicate that they are not used to their full potential in mathematics (Ofsted, 2008), most likely because they are "predominantly teacher-led and mainly focused on presentational software" (JMC, 2011, p.6). This implies that digital technologies are mostly used for demonstrations, acting only as the tutor, which is often how teachers use DGS (Ruthven, 2012; deVilliers, 2006), indicating that there is a big divide between what the software is capable of doing and how even successful mathematics teachers use it (Glover *et al.*, 2007). Falbel (1991) believes technology should not just aid passive learning but help it to be more active, however its use as an explorative tool is not commonplace: "What we do not have is embedded practice in engaging students in using their skills with digital technologies to find out about, learn, apply and communicate aspects of mathematics" (JMC, 2011, p.16). In this article I will concentrate on the following opportunities using DGS: learning from feedback, observing patterns and seeing connections (Becta, 2009), to support concept development in a way which would not be possible with pencil and paper.

Leung (2011) refers to the importance of *techno-pedagogic task design* which "focuses on pedagogical processes in which learners are empowered with amplified abilities to explore, re-construct (or re-invent) and explain mathematical concepts using tools embedded in a technology-rich environment" (p.327). Leung and Bolite-Frant (2015) state that "a multi-tool teaching and learning environment provides learners a milieu where they can interact with different tools and representations" (p.221). Furthermore, Leung and Bolite-Frant (2015) list four key considerations in developing tool-based tasks:

> 1. Use strategic feedback from a tool-based environment to create learning opportunities for student.
>
> 2. Design activities to mediate between the phenomena created by a tool and the intended mathematical concept to be learned.
>
> 3. Make use of the affordances and constraints of a tool to design learning opportunities.
>
> 4. Switch between different mathematical representations or tools. (Leung, 2017, p.73)

Jones (2011), proposes three approaches to DGS pedagogy, illustrated in Figure 1, and Ruthven (2012) describes three case studies which support Jones' framework.

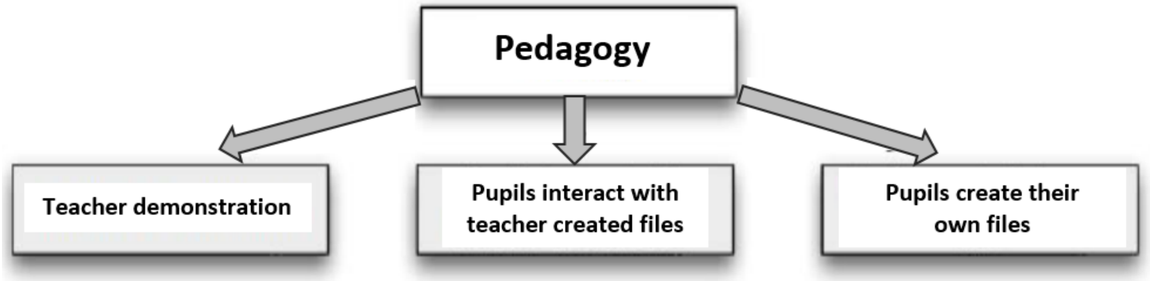

Figure 1: *Framework of teaching approaches with DGS (adapted from Jones, 2011, p.42).*

Both researchers suggest that learners creating their own files is preferable, however Jones (2011) concedes that in the middle scenario, "there is quite some teacher control over the material, but the approach can bring in opportunities for creative thinking and problem solving by learners" (p.42). In fact, according to Maymon-Erez and Yerushalmy (2006), investigation without guidance makes constructing new knowledge less likely, suggesting a minimally pre-constructed file where learners can create as well as manipulate could be the optimal approach (see Appendices 1-3 for more information, available alongside the online version version of this article).

Many researchers have discussed the benefits of dragging in DGS (Hölzl, 1996; Jones, 2002; Baccaglini-Frank & Mariotti, 2010), which aids "the making and justifying of generalisations based on the facility to look at sufficient cases" (Becta, 2009, p.4). It serves as both an amplifier to create an array of examples which would take too long to construct by hand, but also as a reorganiser to see between the examples (Pea, 1985), which many teachers do not appreciate as its key benefit (Ruthven *et al.*, 2008). Dragging also "helps pupils to notice 'what changes and what remains the same' and enables them to formulate and test their conjectures" (Becta, 2009, p.7); however, without guidance, mental constructions are not guaranteed as "understanding dragging means understanding that the dynamic manipulation preserves the critical attributes" (Maymon-Erez & Yerushalmy, 2006, p.5). The fixed attributes in the classroom based project were the properties of the transformations, so it was important for pre-constructed files to allow the learner to perform these rather than just manipulate the images created.

The measurement facility also serves as both an amplifier and a reorganiser, as it is quick and provides a level of accuracy which enables reliable feedback on conjectures. This facility can be used to compare representations, for example comparing the ratios of sides and areas against the scale factor of enlargement, because "a medium which enables pupils to switch effortlessly between these representations enhances their conceptual development" (Becta, 2009). Measurements can be set to a specified degree of accuracy, however Olivero and Robutti (2007) discuss the "double nature of measuring" (p.138), warning that "increasing the number of decimal digits does not mean increasing the precision of the tool" (*ibid*.).

Situated Abstraction

Mathematical abstraction concerns moving from the particular to the general and *decontextualisation* is often viewed as vital in the process of developing higher order thinking (White & Mitchelmore, 2007). Piaget considered three forms of abstraction: empirical, focusing on objects; pseudo-empirical, concerning actions; and reflective, involving mental actions on mental objects (Tall, 2004). Similarly, the cognitive and sociocultural approaches to abstraction focus on moving from concrete to abstract. The cognitive approach achieves this by removing context, while the sociocultural approach considers context and activity to be inseparable due to the learner's personal history (Herschkowitz *et al.*, 2001). However, does the concrete necessarily precede the abstract? The dialectic perspective considers that the process does not start from the concrete at all, rather it shifts "from an undeveloped to a developed form of the abstract" (*ibid*., p.200).

Herschkowitz *et al.* (2001) take the sociocultural perspective, defining abstraction as "an activity of vertically reorganising previously constructed mathematics into a new mathematical structure" (p.195), where learners establish and strengthen interconnections, ensuring mathematical consistency between concepts. They identify observable elements of abstraction: constructing, recognising and building-with. Constructing is the process of building complex structures from simpler ones; recognising is seeing connections with previous knowledge which, according to the authors, is easier to observe than constructing; and building-with is transferring previously constructed knowledge by applying it to different contexts (Herschkowitz *et al.*, 2001). These actions are not completely independent of each other, nor necessarily linear in nature, as when constructing, the learner may need to access recognising and building-with structures from previous situations.

In situated cognition, the classroom culture provides the context in which learning can take place (Brown *et al.*, 1989) and the impact of the environment on learning is pivotal (Nardi & Stewart, 2003; Boaler, 1999). Figure 2 illustrates that where the two theories of situated cognition and mathematical abstraction meet, is another model of abstraction in context: Noss' & Hoyles' (1996) *situated abstraction*, a process of making connections in mathematics through making sense of a concrete situation (Pratt & Noss, 2002). From this definition, situated abstraction supports the notion of microworlds, which are "designed to be discovery-rich in the sense that little nuggets of knowledge have been scattered around in it for you to find" (Papert, 1987, p.80).

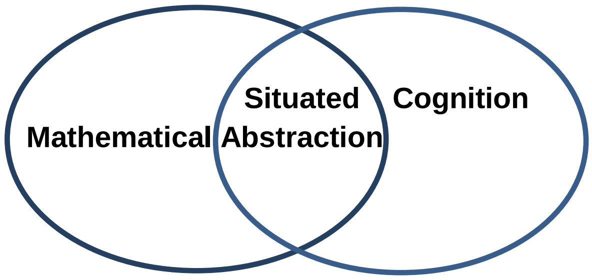

Figure 2: *Combining situated cognition and mathematical abstraction.*

In situated abstraction, instead of simply taking one extreme of learning in situ or the other of complete decontextualisation, which could be argued is not possible as the setting itself provides context, mathematical meaning is constructed by "drawing context into abstraction, populating abstraction with objects and relationships of the setting" (Pratt & Noss, 2010, p.27), suggesting it is broadening the context which allows learners to establish connections (Pratt & Noss, 2002). Pratt and Noss (2010) also view *purpose* as a key component in facilitating abstractions, as a purposeful task should maximise the significance of the feedback given from a microworld, which should lead to learners more readily appreciating the *utility* of the task which they believe is "an essential element of mathematical abstraction" (p.12), supporting Papert's (1972) stance that motivation in mathematics comes from "true personal involvement" (p.251).

Situated abstraction develops the ideas of building knowledge into schemas (Dubinsky & McDonald, 2002) and scaffolding in abstraction (Ozmantar & Roper, 2004), into the use of technology to establish connections, which Noss and Hoyles (1996) call *webbing*. However, according to Herschkowitz *et al.* (2001), Noss and Hoyles were not explicit how webbing linked to constructing new knowledge and so "did not provide a framework within which to investigate the process of abstraction" (p.198). In contrast, Pratt and Noss (2010) argue that situated abstractions are more observable than diSessa's (1993) phenomenological primitives (p-prims), which, in her model of abstraction in context, are small, unstructured fragments of knowledge, abstracted from experience and built into concepts called coordination classes. The classroom based project focuses on the situated abstraction model to explore concept development using DGS, but considers how observing learners constructing, recognising and building-with could help link constructing new knowledge with webbing.

Enlargement, Similarity and Proportional Reasoning

In general, the existing research is very positive about the use of DGS to support the learning of geometry (e.g. Arzarello *et al.*, 2013; Jones, 2011; Ruthven *et al.*, 2008), even if research on transformational geometry is limited (Jones, 2012). So why is it that mathematics teachers are not using it effectively? It could be argued that it is the examination system which is restricting teachers' use of DGS in the classroom as with so many assessment criteria, teachers may not feel they have the time to dedicate a series of lessons to investigating scale factor when enlargement may only be worth a few marks in the learners' final exam. However, examination boards in the UK have reported weaknesses in learners' understanding of scale factor (e.g. OCR, 2012), with some candidates using different scale factors on each dimension of the shape given (Edexcel, 2012). The most common incorrect scale factor used is 2 (*ibid*.), which could be because traditional methods use this much more than others (e.g. Payne *et al.*, 2006), possibly because learners are restricted by space to perform large scale factors by hand. There is also a lack of understanding that fractional scale factors produce a reduction (Edexcel, 2012) and that negative scale factors do not (OCR, 2012). Some learners do not even appreciate that scale factor is multiplicative and they simply add it to each side (Edexcel, 2010; 2011) and the language used by learners to describe an enlargement will often not involve the term scale factor at all (Edexcel, 2010; OCR, 2011), perhaps as a result of traditional methods not allowing for rich discussion. Issues surrounding the role of the centre of enlargement are exposed when learners describe enlargements, as many describe an enlargement followed by a translation (OCR, 2011), suggesting no understanding of the effect of the centre on the position of the image. It is not surprising that with these misconceptions regarding simple enlargement, learners continue to struggle with the concept of similarity (Edexcel, 2016), for example "few could explain why two rectangles were **not** similar and almost none mentioned scale factors or ratios" (OCR, 2015, p.8). It is also common for learners to apply the linear scale factor to areas or angles, even if the solution is unreasonable (OCR, 2012) and "the understanding of area and volume scale factors continues to elude many" (OCR, 2015, p.19).

Proportional reasoning "is both the capstone of elementary arithmetic and the cornerstone of all that is to follow" (Lesh *et al.*, 1988, p.94) and appreciation of scale factor connects number and geometry (Cox, 2013), therefore understanding this concept in its entirety should support learners to solve enlargement and similarity problems as well as solving other problems involving ratios (e.g. map scales, best buys, proportional division, trigonometry). These connections however are often not explicit in textbooks (Lamon, 1995), which often devote entirely separate chapters to each (see Baston *et al.* 2010; Payne *et al.*, 2006; Banks & Alcorn, 2003). Dragging and measurement with DGS could help address issues with transformations (Hollebrands, 2003), facilitating discussion and enabling generalisations and connections to be made. However, assessment in the UK has been a barrier to teaching with technology (JMC, 2011), something which has not seemed to change with the current GCSE 9-1 specifications, where only one digital technology is alluded to: the calculator (Edexcel, 2015; AQA, 2014).

Many researchers, including Piaget, have described proportional reasoning as a global ability (Lesh *et al.*, 1988), implying it is a general cognitive strategy, transferable to any situation. However, as learners are often not consistent in their level of reasoning, Lesh *et al.* (1988) argue that it is a more local competence which is "initially mastered in small and restricted classes of problem settings" (p.102). Exploring enlargement and similarity within the set parameters of a DGS microworld could support learners to test out their own conjectures in a mathematically consistent space, allowing the construction of new knowledge locally from the multitude of examples given through dragging. The measurement facility could then support learners to recognise and build-with their prior knowledge of ratio, adding to their proportional reasoning *web*, with the intention of eventually leading to more global problem solving.

THE CLASSROOM BASED PROJECT

Research Design and Methodology

A small scale classroom based project was conducted on enlargement and similarity using Geometer's Sketchpad (GSP) as the dynamic geometry software, with the aim of improving this area of teaching in my school at the time. The following research question was investigated:

How can dynamic geometry software enhance and strengthen the teaching of transformational geometry, in particular in making connections between enlargement and similarity?

The project was conducted by means of a case study approach to allow for depth of study into the learners' mathematical thinking and how this learning might occur; however, this approach has the limitation that any findings could be unique to the case study group (Denscombe, 2007). The group selected consisted of 31 girls aged 13-14 from an 11-16 girls' school where the proportion of students who are from minority ethnic backgrounds or speak English as an additional language (EAL) is above average. The school is non-selective and located on the south coast of England in a town where two out of the 10 secondary schools are grammars. In order to be as representative as possible of the whole school, purposive sampling was used to select this class, as the girls had a similar proportion of EAL, were the median age group in the school and had an average level of attainment in mathematics. There are limitations to this strategy including: the small size of the group; from only one school and hence the same geographical area; the fact that the setting was single-sexed; and there was no control group to give quantifiable evidence that any conceptual improvements were as a result of the DGS. Although it is not possible to generalise from this case study (Denscombe, 2007), due to the purposive sampling from within the school, it was intended that any findings could be used to inform planning across the school's mathematics department. The context of the school also provides the reader the opportunity for comparison with similar schools.

The class worked in pairs on transformation geometry for six lessons, three of which were in the computer room:

Lesson 1: Recap of how to perform reflections, rotations and enlargement using traditional pencil and paper methods

Lesson 2: Transforming Dance activity on GSP (see Appendix 1, available online)

Lesson 3: Recap on enlargement by a positive integer scale factor using traditional pen and paper methods

Lessons 4 and 5: Exploring enlargement and similarity on GSP (see Appendix 3, available online)

Lesson 6: Solving similarity problems in the classroom

In part the programme of lessons was dictated by access to the school's computer rooms, however the learners had previously met pen and paper methods for simple transformations so the earlier classroom based lessons were an opportunity to remind learners of these methods before seeing how working with GSP could enhance and extend that prior learning.

Responses to the GSP activities were recorded on task sheets by the students and some of their discussions were observed first hand by the author. These discussions were recorded by scribing the dialogue observed while at the same time questioning learners to help move learning forward. There is the limitation that I both facilitated the activities and observed the interactions (BERA, 2011), which could have an effect on the validity and objectivity of the data collected, however the findings are only intended as a means of improving the learning environment in this school rather than an attempt to generalise beyond this. To follow the *Ethical Guidelines for Educational Research* (*ibid*.), the class were informed of the purpose of the project and how their written responses and scribed discussions would be used to form my research. Permission was obtained in loco-parentis from the Head Teacher to conduct the study. Furthermore, to maintain confidentiality and anonymity, the students referred to in this report are done so by initials only (*ibid*.).

Getting to Grips with GSP

The learners used a desktop version of GSP, however since the classroom based project, which was conducted in 2012, touchscreen technology has developed widespread usage. The learners had previously only seen the software used on an interactive whiteboard, which ironically relates more closely to how DGS is used on handheld touchscreen technology, although the latter allows for manipulation with both hands simultaneously (Arzarello *et al.*, 2013), so learning the basic functions needed to be incorporated into the activities. This was achieved by using GSP to explore combinations of transformations after performing reflections, rotations and translations on paper the previous lesson. The activity designed for this was entitled 'Transforming Dance' (see Appendix 1, available alongside the online version of this article), adapted from a task by Olive (2000) in which learners produce stickmen with GSP and then, using the transformation functions, make their stickmen dance. To address purpose, a physically active dancing 'starter' was devised using translations, reflections and rotations, which may have increased motivation in the GSP task as students seemed to enjoy recreating the dance moves with their stickmen. The task prompted learners to discuss the effects of combining transformations on their stickmen, with the intention of constructing new knowledge. This was then extended with an activity on performing multiple transformations on polygons and describing their equivalent single transformations (see Lack, 2011), with the intention of enabling learners to recognise how they needed to apply the knowledge from the stickmen activity, providing the utility of the task (Ainley *et al.*, 2006).

I noted several differences between the introduction to GSP lesson and performing transformations by hand the previous lesson. Firstly, students were generating and testing conjectures in just one session, perhaps because GSP acted as an amplifier, allowing students to perform combinations of transformations at a speed which on paper would not have been achievable with this class. Secondly, I observed that AF, a particularly artistic student, was more engaged in the lesson than usual, creating intricate figures which would have been too time consuming, and perhaps too difficult, to transform on paper. Papert (1971) argued that microworlds aid pupil involvement; for AF her level of engagement and creative response to the task could indicate that the dynamic geometry environment acted as a motivational aide. GSP also provided more accuracy than many could achieve on paper, which was particularly true for GB, who the previous lesson had struggled with drawing transformations by hand, was able to produce quick and accurate diagrams. This could have supported learning for GB as she was able to use her accurate results to describe the effect of double reflections in parallel lines. Finally, the dragging facility allowed students to physically move the original object to see the changing effect on the image, allowing them to see many examples as opposed to just one. The conjectures students made, on the effect of double reflections for example, could be attributed to this movement away from specific concrete examples, indicating that GSP acted as a reorganiser for performing transformations, which may have allowed more abstract generalisations to be made.

According to Pratt and Noss (2010) it is important to design tasks which allow for abstraction. During the introductory task, students may have primarily made abstractions as a result of the dragging feature allowing them to move from the particular to the general, so it was important to incorporate this into the main task on enlargement and similarity. I also decided to see how the measurement facility could support learners to conceive abstractions by establishing connections between scale factor and ratio. I designed four activities (see Appendix 2, available alongside the online version of this article) for the class to work on in pairs over two lessons:

1. Exploring enlarging a triangle from a given centre
2. Exploring corresponding angles in similar triangles

3. Exploring corresponding sides in similar triangles
4. Exploring areas of similar triangles

The activities were piloted prior to the lessons with two students, JC and SA, and as a result modifications were made (see Appendix 3, available alongside the online version of this article) before carrying out the task with the rest of the group. JC and SA then acted as *Lead Learners* (Stewart, 2009), using their new knowledge to support the rest of the class, affording me time to scribe interactions between the pairs. Herschkowitz *et al.* (2001) believe that the construction of knowledge is most likely to be observed in a social setting, so to facilitate the observations of possible abstractions in this project, each task was designed to prompt peer-discussion. The tasks were also designed to encourage explanation, justification and reasoning which is considered important when designing tasks using GSP (Jones, 2011) and indeed in the learning of mathematics in general. Initially there was only one context, the task itself in the dynamic geometry environment, however Pratt & Noss (2002) discuss the importance of applying knowledge in different situations, so following these sessions, students solved enlargement and similarity problems in the classroom to see if they could apply their knowledge.

Dragging from the Particular to the General

In the pilot task, the following dialogue ensued when discussing the effect of dragging the centre of enlargement:

> SA: When you move the point further away, will the triangle follow?
> JC: No, it will end up further over there…yes I'm right.
> SA: Oh, because the distance is further away it ends up further away in the other direction.
> Teacher: What happens if you move the centre to the edge or inside the original?
> SA: The copied image is spread around.
> JC: The distances are still doubled.

SA started the session unaware of the effect of the centre of enlargement, however after this activity, and with support from JC, she could see the how the centre dictated the position of the image, this could have been supported by dragging, as she was able to move from the specific example they started with to a whole set of examples. After the activity both were able to correctly predict the position of the image when I pointed to possible centres, indicating an abstraction may have been made. A similar discussion was observed between AO and RT in the whole class session:

> RT: If I move the point of enlargement…
> AO: The closer…oh, it overlaps with the original!
> Teacher: What effect is the centre having?
> RT: Changes where the new one is.
> AO: If the point is far away the new one is far away too and if closer it overlaps.
> Teacher: If the centre of enlargement was above the original triangle, where would the image be?
> AO: Here [points below]
> Teacher: And if the centre was to the right?
> AO: On the left.
> RT: So if here [points to left side] would be here [points to right]. Let's check!

On analysing written responses to the activity, all students discovered that the centre of enlargement affected the position of the image and most could describe roughly where the image would be depending on that position. Literature shows that feedback from DGS "is not only quick, but also reliable, non-judgemental and impartial" (Becta, 2009, p.2). AO and RT seemed to value this feedback and appeared to gain in confidence in their understanding of enlargement, making generalisations just ten minutes into the activity. Papert (1987) describes a microworld as an environment which is "created and designed as a safe place for exploring" (p.80), in this respect the use of GSP in this project could be considered a microworld.

A pre-constructed file using a slider was created to investigate the effect of scale factor to support seeing the general as well as the particular, which could be less likely if you type in specific scale factors into the dialogue box:

> JC: Move the slider to 3
> SA: It's not equal [points to distances between the centre and corresponding vertices]
> JC: That's because it's twice the distance here and once here so three times in total. Scale factor 3.
> SA: Ah, ok.
> Teacher: What do you think would happen for scale factor 4?
> SA: Will that be three times the distance [points between corresponding vertices] so four times in total?
> Teacher: And what can you say about the sides?
> SA: They're four times bigger
> Teacher: What happens if the scale factor is less than one?
> SA: Move the slider to ½
> JC: The original triangle is the enlargement and the image is the original if that makes sense!
> SA: Instead of enlarging, it shrunk!
> JC: It's de-larged…is that a word?
> Teacher: Has it reduced in size?
> JC: Yes reduced!

The language JC and SA were using had evolved since the start of the session, using the terms image and scale factor confidently, perhaps as a result of rich discussion. They seemed to recognise the general effect of the scale factor and appeared to construct scale factors less than one into their 'web' of knowledge, perhaps building a new, more complex structure from the simpler structure of positive integer scale factors. Their initial conjecture was that the triangles had swapped positions, however through their discussion and further dragging they reorganised their knowledge of scale factors in order to deal with multiplying by a proper fraction. Due to the new language developed and the construction made, this could be considered an abstraction (Herschkowitz *et al.*, 2001). JC and SA even accidentally discovered negative scale factors when the slider moved past zero which they also attempted to connect to their scale factors web in response to the new knowledge:

> JC: It's flipped!
> SA: Wow!
> JC: So if you change it to -1 it would be exactly the same distance apart [as +1] but over there. What about -2?

Perhaps as a result of dragging, a deeper understanding of scale factor had been developed and GSP seemed to spark a real interest, which could be why they were able to make higher level conjectures than their current level of attainment. AO and RT were also observed discovering negatives, attempting to generalise in the same way as for positive scale factors and on analysis of the written responses, so had other pairs despite not being prompted. The class had not been expected to deal with negative scale factors until GCSE, so dynamic geometry could have allowed them to make abstractions beyond their supposed ability, agreeing with Ruthven (2012) that unexpected results can be a positive element of DGS. For CM and SS, dragging the slider from 2 to ½ also gave significance to scale factor 1:

> CM: It gets smaller. If you did it at one it'd be exactly the same.
> SS: So you're like halving it. When it's smaller than one you halve it, when at one it stays the same.
> CM: You only halve it for scale factor ½

Measurement and Accuracy

When exploring corresponding angles, CL and MH started with the misconception that scale factor also applies to angles; however the angle measurement facility gave the pair immediate feedback from this incorrect conjecture and some additional support from me appeared to allow them to readjust their thinking and further develop the concept:

> CL: What happens to the angles?
> MH: They're getting bigger
> Teacher: All of them? What if you compare corresponding angles? [Indicates on screen]
> [MH measures angles]
> Teacher: What do you know about angles in a triangle?
> CL: They add up to 180°.
> Teacher: So can all the angles get bigger when you enlarge?
> MH: No they get smaller. No, stay the same [laughs]. You're not changing the angles, it's the edges getting longer. You can't double the angles. You'd get 360 which is a circle.
> Teacher: Which shape has interior angles adding to 360°?
> MH: Oh, a square!

In past GCSE examinations, "better candidates instinctively knew that enlargement preserved angle but far too many multiplied by the scale factor to get an unreasonable angle" (OCR, 2012, p.20). On analysis of the written responses, most pairs appreciated that the scale factor had no effect on the angle size, which could be attributed to the angle measurement facility and the ability to drag; only one pair had not understood that angles are preserved. This indicates that these students could have conceived an abstraction using DGS that many learners do not make by the end of secondary school. In addition, it was observed that using GSP with support from the teacher, helped one pair of students (PW and LT) understand angle notation, which at the start of the session they were not secure with, as they needed to be shown how to click on the vertices in the correct order to measure the required angle.

Clements (2002) highlights the importance of the role of the teacher to ensure learners see the mathematics and make the abstractions intended. Both the conversation with CL and MH above and the support needed for PW and LT demonstrate this importance; however, with a class of 31 students, time for such a rich discussion with every learner is limited. Research has shown that managing time during lessons with DGS needs development (Jones, 2011) and this was certainly found to be the case with this project, even with the addition of two lead learners.

Although there is a facility for calculating ratios of sides in GSP, using the calculator function instead seemed to reinforce the multiplicative nature of scale factor and the importance of corresponding sides, which could be overlooked by learners if the process of this calculation is removed:

> SA: The ratio is the same as the scale factor
> JC: Only if you do the right sides. If you do k' divided by j it won't as the sides are different.

For JC and SA, the measurement facility may have both amplified and reorganised the exploration of corresponding sides in similar triangles, enabling this comparison of ratio of sides to scale factor.

Olivero and Robutti (2007) warned that although helpful to aid abstractions, accuracy in measurement using GSP could also be an issue. Here, the activity had been designed for abstraction using a slider to enable students to see the general rather than just amplifying the particular, however when it came to comparing area scale factors, the slider proved to be a problem as although it displayed 2.00 for the scale factor, GSP stores numbers more accuracy than it displays (*ibid*.), so when JC and SA calculated the area ratio they got 3.98 instead of 4.00. This made it difficult to spot patterns and so the class were advised to change their area ratios to be correct to the nearest unit. When students tabulated these results, most students were able to quickly spot the connection between the length ratio and the area ratio and JC appeared to make connections with squaring and area, building on her web of knowledge on proportional reasoning:

> SA: Oh I get it, it will be 36. It's squaring it.
> JC: Yes, because it's area.

Transferable Abstractions

The class were asked to anonymously complete a feedback questionnaire (see Appendix 4, available alongside the online version of this article), to allow students to be honest about their experiences using GSP (Denscombe, 2007). The feedback was generally very positive, with students indicating that they found the activities enjoyable and interesting with most students feeling they had progressed in their understanding of enlargement. The negative comments focused on difficulties using the software, perhaps supporting Jones' (2011) assertion that "even with carefully designed tasks, sensitive teacher input, and a classroom environment that encourages conjecturing and a focus on mathematical explanation, it can take quite some time for the benefits of using DGS to emerge" (p.40).

The following lesson the transferable nature of any abstractions they may have made was investigated through two activities: a sorting activity (see French, 2012) where students matched similar triangles and their scale factors, which demonstrated that students could apply their knowledge of preserved angle by identifying pairs of similar triangles and most could then calculate the scale factors and missing lengths; the class then completed a 'rally coach' activity, where students took turns to support each other to solve similarity problems. In the written responses, all students could find linear scale factor and use it to find missing lengths, suggesting an understanding of the multiplicative nature of scale factor. The area scale factor proved to be too challenging for some who needed prompting that the linear scale factor did not apply to area, this could however be attributed to lack of time spent on the last GSP activity rather that the potential of DGS to support this concept.

CONCLUSIONS

Dynamic geometry can act as a microworld to support concept development, which is achieved through the dragging and measurement facilities acting as both amplifiers, speeding up processes and as reorganisers, moving the learner from the particular to the general, in a way that is not achievable using pencil and paper alone. As evidenced by the literature review, these factors, along with others such as non-threatening, accurate feedback and peer-discussion, can facilitate pattern spotting and making connections, enabling situated abstractions more likely to be conceived. DGS can increase confidence, motivation and understanding when managed effectively by the teacher, however this is the element that needs greatest development to ensure the potential of dynamic geometry is realised, therefore further research is needed into strategies teachers can use to manage the learning of large classes using DGS, which was found to be an obstacle in my empirical research.

The findings from the classroom based project, although small-scale, support the existing literature in terms of the amplification and reorganisation effects of both dragging and measurement in DGS. Furthermore, the findings suggest that teaching transformations with GSP has great potential to improve learners' ability to make and test conjectures in transformation geometry and to make connections between scale factor and ratio. The literature review suggests proportional reasoning is local competence, which through constructing and webbing knowledge can lead to more global problem solving. To make these situated abstractions more likely to occur, a combination of designing minimally pre-constructed tasks, peer-discussion and utilising the dragging and measurement facilities are paramount to allow movement from the concrete to the abstract.

REFERENCES

Ainley, J., Pratt, D., and Hansen, A. (2006), Connecting engagement and focus in pedagogic task design, *British Educational Research Journal*, 32(1), 23-38

Arzarello, F., Bairral, M., Danie, C., & Yasuyuki, I. (2013), Ways of manipulation touchscreen in one geometrical dynamic software. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching* (pp. 59–64), Bari: University of Bari.

AQA (2014), GCSE Mathematics (8300), Manchester: AQA Education, www.aqa.org.uk/8300, accessed 26 May 2017

Baccaglini-Frank, A. and Mariotti, M. (2010), Generating Conjectures in Dynamic Geometry: The Maintaining Dragging Model, International Journal of Computers for Mathematical Learning *15*, 225–253

Banks, T. and Alcorn, A. (2003), *Mathematics for AQA GCSE – Higher Tier*, Lancashire: Causeway Press Limited

Baston, C., Bolter, J., Cole, G., Dyer, G., Flowers, M., Hughes, K., Jolly, P., Knott, J., Linsky, J., Newman, G., Pepper, R., Petran, J., Pledger, K., Summerson, R., Tanner, K. and Western, B. (2010), *Mathematics A - Linear Foundation*, Harlow: Edexcel Limited

Becta (2009), *Secondary Mathematics with ICT: A pupil's entitlement to ICT in secondary mathematics,* http://www.nationalstemcentre.org.uk/dl/2a99cf80d5efac70875df99f983cfac1de333a72/13810-entitlement_sec_maths.pdf, accessed 13 April 2013

BERA (2011), *Ethical Guidelines for Educational Research,* London: British Educational Research Association

Boaler, J. (1999), Participation, knowledge and beliefs: A community perspective on mathematics learning, *Educational Studies in Mathematics*, 40(3), 259-281

Brown, J. S., Collins, A. and Duguid, P. (1989), Situated Cognition and the Culture of Learning, *Educational Researcher,* 18(1), 32-42

Clements, D.H. (2002), Computers in Early Childhood Mathematics, *Contemporary Issues in Early Childhood*, 3(2), 160-181

Cox, D. (2013), Similarity in Middle School Mathematics: At the Crossroads of Geometry and Number, *Mathematical Thinking and Learning*, 15(1), 3-23

Denscombe, M. (2007), *The Good Research Guide for small-scale social research projects* (third ed.), Maidenhead: Oxford University Press (first published 1998)

deVilliers, M. (2006), Some Pitfalls of Dynamic Geometry Software, *Learning and Teaching Mathematics*, 4, 46-52

diSessa, A. (1993), Toward an Epistemology of Physics, *Cognition and Instruction*, 10(2-3), 105-225

Dubinsky, E. and McDonald, M.A. (2002), APOS: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research, in Hoton, D. (ed.), *The Teaching and Learning of Mathematics at University Level*, New ICMI Study Series, 2002, Vol. 7, Section 3, pp.275-282.

Edexcel (2016), *Principal Examiner Feedback Summer 2016 1MA0_1F_1606_ER*, Pearson Education Ltd

Edexcel (2015), Level 1/Level 2 GCSE (9–1) in Mathematics Specification – Issue 2, Pearson Education Ltd

Edexcel (2012), *Principal Examiner Feedback Summer 2012 GCSE Mathematics (Linear) 1MA0*, Pearson Education Ltd

Edexcel (2011), *Examiner Report June 2011 GCSE Mathematics (1380),* Edexcel Ltd, Publications Code UG028357

Edexcel (2010), *Principal Examiner Feedback Summer 2010 GCSE Mathematics (1380)*, Edexcel Ltd

French (2012), *Similar Triangle Match Up*, http://www.tes.co.uk/teaching-resource/Similar-Triangle-Match-Up-6185944/, accessed 24 March 2013

Glover, D., Miller, D., Averis, D. and Door, V. (2007), The evolution of an effective pedagogy for teachers using the interactive whiteboard in mathematics and modern languages: an empirical analysis from the secondary sector, *Learning, Media and Technology*, 32(1), 5-20

Hershkowitz, R., Schwarz, B. and Dreyfus, T. (2001), Abstraction in Context: Epistemic Actions, *Journal for Research in Mathematics Education*, 32(2), 195-222

Hollebrands, K. (2003), High school students' understandings of geometric transformations in the context of a technological environment, *Journal of Mathematical Behavior 22*, 55–72

Hölzl, R. (1996), How does 'dragging' affect the learning of geometry, *International Journal of Computers for Mathematical Learning* , 1(2), 169-187

JMC (2011), *Digital technologies and mathematics education*, Clark-Wilson, A., Oldknow, A. and Sutherland, R. (eds.), Joint Mathematical Council of the United Kingdom *(JMC)* http://cme.open.ac.uk/cme/JMC/Digital%20Technologies%20files/JMC_Digital_Technologies_Report_2011.pdf, accessed 26 March 2013

Jones, K. (2012), Using Dynamic Geometry Software in Mathematics Teaching, *Mathematics Teaching 229*, ATM

Jones, K. (2011), The value of learning geometry with ICT: lessons from innovative educational research, in Oldknow, A. and Knights, C. (eds.), *Mathematics Education with Digital Technology*, London: Continuum, pp.39-45

Jones, K. (2002), Research on the use of dynamic geometry software: implications for the classroom, *MicroMath*, 18(3), 18-20

Lack (2011), *Exploring combined transformations group activity,* http://www.tes.co.uk/teaching-resource/GCSE-Maths-Game-Transformations-Lesson-6140178/, accessed 9 March 2013

Lamon, S. (1995), Ratio and Proportion: Elementary Didactical Phenomenology, in Sowder, J. and Schappelle, B. (eds.), *Providing a foundation for teaching mathematics in the middle grades*, Albany: SUNY Press

Lesh, R., Post, T. and Behr, M. (1988), Proportional reasoning, *Number concepts and operations in the middle grades*, 2, 93-118

Leung, A. (2017), Variation in Tool-Based Mathematics Pedagogy. In *Teaching and Learning Mathematics through Variation* (pp. 69-84), Sense Publishers.

Leung, A. (2011), An epistemic model of task design in dynamic geometry environment, *ZDM*, 43(3), 325-336.

Leung, A. and Bolite-Frant, J. (2015), Designing mathematics tasks: The role of tools. In A. Watson and M. Ohtani (Eds.), *Task design in mathematics education: The 22nd ICMI study* (pp. 191–225), New York, NY: Springer

Maymon-Erez, M. and Yerushalmy, M. (2006), "If You Can Turn a Rectangle into a Square, You Can Turn a Square into a Rectangle..." Young Students Experience the Dragging Tool, *International Journal of Computers for Mathematical Learning*, 11(3), 271-299

Nardi, E. and Steward, S. (2003), Is Mathematics T.I.R.E.D.? A Profile of Quiet Disaffection in the Secondary Mathematics Classroom, *British Educational Research Journal*, 29(3), 345–367

Noss, R. and Hoyles, C. (1996), *Windows on Mathematical Meanings: Learning Cultures and Computers*, Volume 17, London: Kluwer Academic Publishers

OCR (2015), *General Certificate of Secondary Education Mathematics A (J562) OCR Report to Centres,* Cambridge: Oxford Cambridge and RSA Examinations

OCR (2012), *General Certificate of Secondary Education Mathematics A (J562) OCR Report to Centres*, Cambridge: Oxford Cambridge and RSA Publications

OCR (2011), *General Certificate of Secondary Education Mathematics A (J562) Examiners' Reports*, Nottingham: Oxford Cambridge and RSA Publications

Ofsted (2008), *Mathematics: understanding the score*, London: HMSO.

Olive, J. (2000), Implications of Using Dynamic Geometry Technology for Teaching and Learning, *Teaching and Learning Problems in Geometry* conference, Fundão, Portugal, 6-9 May 2000, http://math.coe.uga.edu/Olive/Portugal/Portugal_paper.html, accessed 9 March 2013

Olivero, F. and Robutti, O. (2007), Measuring in dynamic geometry environments as a tool for conjecturing and proving, International Journal of Computers for Mathematical Learning, 12(2), 135-156

Ozmantar, M. F. and Roper, T. (2004), Mathematical abstraction through scaffolding*, Proceedings of the 28th Conference of the International,* 3, 481-488

Papert, S. (1991), Situating Constructionism, in Papert, S. and Harel, I (eds), *Constructionism*, Norwood, NJ: Ablex Publishing Corporation

Papert, S. (1987), Microworlds: transforming education, *Artificial intelligence and education*, 1, 79-94

Papert, S. (1972), Teaching children to be mathematicians versus teaching about mathematic', *International Journal of Mathematical Education in Science and Technology*, 3(3), 249-262

Payne, G., Holt, R., Rayment, M. and Robinson, I. (2006), *AQA GCSE Mathematics – Higher*, Oxford: Heinemann

Pea, R. D. (1985), Beyond amplification: Using the computer to reorganize mental functioning, *Educational Psychologist*, 20(4), 167-182

Pratt, D. and Noss, R. (2002), The Microevolution of Mathematical Knowledge: The Case of Randomness, *The Journal of the Learning Sciences*, 11(4), 455–488

Pratt, D. and Noss, R. (2010), Designing for Mathematical Abstraction, *International Journal of Computers for Mathematical Learning*, 15(2), 81-97

Rayner, D. (2003), *Modular Mathematics Higher GCSE for AQA*, Oxford: Oxford University Press

Ruthven, K., Hennessy, S. and Deaney, R. (2008), Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice, *Computers & Education*, 51, 297–317

Ruthven, K. (2012), The didactical tetrahedron as a heuristic for analysing the incorporation of digital technologies into classroom practice in support of investigative approaches to teaching mathematics, ZDM *,* 44(5), 627-640

Stewart, W. (2009), Pupils take hold of the teaching**,** *TES*2 January 2009**,** http://www.tes.co.uk/article.aspx?storycode=6006663, accessed 14 April 2013

Tall, D. (2004), Building Theories: The Three Worlds of Mathematics, *For the Learning of Mathematics*, 24(1), 29-32.

Taylor, R. P. (1980), *The computer in school: Tutor, tool, tutee*, New York: Teachers College Press

White, P. and Mitchelmore, M. (2007), Conceptual Knowledge in Introductory Calculus, *Mathematics Education Research Journal*, 19(2), 1-9